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q-Power function over q-commuting variables and deformed XXX, XXZ chains

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Abstract

We find certain functional identities for the Gauss q-power function of a sum of q-commuting variables. Then we use these identities to obtain two-parameter twists of the quantum affine algebra $U_q(\widehat{sl}_2)$ and of the Yangian $Y(sl_2)$. We determine the corresponding deformed trigonometric and rational quantum R-matrices, which then are used in the computation of deformed XXX and XXZ Hamiltonians.

1 Introduction

The most famous R-matrices, found by Yang, Baxter and Zamolodchikov, satisfy the Yang-Baxter (YB) equation due to addition laws for basic rational, trigonometric and elliptic functions. This note is an attempt to answer the following question: which elementary functions and which of their properties could be employed to produce other solutions of the YB equation.

There is a general opinion, that all the solutions of the Yang-Baxter equation, as well as the corresponding Hopf algebras, can be obtained from the Drinfeld-Jimbo solutions by suitable twists. Recently, all finite-dimensional bialgebras from the Belavin-Drinfeld list

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[2] were quantized in this way [4]. The first nontrivial infinite-dimensional examples, which cannot be reduced to the finite-dimensional case, are a classical rational and a trigonometric r-matrix with values in sl_2, found in [2, 10]. They can be obtained from the classical Yang and Drinfeld-Jimbo r-matrices by adding respectively a certain (but the same!) polynomial of the first degree in the spectral parameters. We found the corresponding twist for the Yangian $Y(sl_2)$ and extended it to a two-parameter twist of the quantum affine algebra $U_q(\widehat{sl}_2)$.

Surprisingly, it has the simple form of a q-power function, but with q-commuting arguments, its Yangian degeneration becomes the usual power function whose arguments belong to an additive variant of the Manin q-plane. In this setting (q)-power functions satisfy nontrivial generalizations of their standard properties (see eqs. (9)-(11)), which guarantee the cocycle identity for the twists.

We calculate the corresponding deformations of the traditional trigonometric and rational R-matrices, putting them into a single family and compute the related Hamiltonians of the periodic chains. It gives two-parameter integrable deformations of the XXZ and XXX Heisenberg chains. As a particular case we get the deformed XXX chain treated in [9].

2 q-power function over q-commuting variables

Denote by $(1-u)_q^{(a)}$ the following q-binomial series [5]:

$$F_a(u) = (1-u)_q^{(a)} = 1 + \sum_{k>0} \frac{(-a)_q(-a+1)_q \cdots (-a+k-1)_q}{(k)_q!} u^n.$$

Here $(a)_q = \frac{q^a-1}{q-1}$. This unital formal power series over u satisfy the following additive properties:

$$(1-u)_q^{(a)}(1-q^{-a}u)_q^{(b)} = (1-u)_q^{(a+b)}, \tag{1}$$

$$(1-u)_q^{(a)}(1-v)_q^{(a)} = (1-u-v+q^{-a}uv))_q^{(a)}, \tag{2}$$

$$(1-v)_q^{(a)}(1-u)_q^{(a)} = (1-u-v+uv))_q^{(a)} \tag{3}$$

where the variables v and u in (2) and in (3) q-commute: $vu = quv$, and is uniquely characterized by the difference equation

$$F_a(u) = \frac{1 - q^{-a}u}{1 - u} F_a(qu) \tag{4}$$

which follows directly from the definition. The relation (1) can be checked directly on the level of formal power series. All the other properties can be deduced from the presentation of the q-power function as a ratio of q-exponential functions and from the corresponding properties of q-exponents:

$$(1 - u)_q^{(a)} = \frac{\exp_q \frac{u}{1-q}}{\exp_q \frac{uq^{-a}}{1-q}} = \frac{(q^{-a}u; q)_\infty}{(u; q)_\infty}. \tag{5}$$

Here

$$\exp_q(u) = 1 + \sum_{k>0} \frac{u^n}{(n)_q!} \qquad\qquad (u; q)_\infty = (1 - u)(1 - qu) \cdots$$

To prove the relation (5), one can note that both sides satisfy the same difference equation (4) under assumption $|q| < 1$. Clearly, under this assumption, the solution $F_a(u)$ of (4) is unique if $F_a(0) = 1$. Thus both sides of the first equality in (5) coincide as formal power series. Then the relation (1) is a direct corollary of (5), while (2) and (3) follow from the addition law [8] for q-exponents (6) and from the Faddeev-Volkov [6] identity (7), where again $vu = quv$:

$$\exp_q(u) \exp_q(v) = \exp_q(u + v) \tag{6}$$

$$exp_q(v) \exp_q(u) = \exp_q(u + v + (q - 1)vu) \tag{7}$$

We refer to (2) and (3) also as Faddeev-Volkov identities. Below we will give a different proof of a more general relation and get (2) and (3) as its consequences.

Let us consider now the q-power series as a function of a sum of two q-commuting variables u and v, $vu = quv$:

$$F_a(u + v) = (1 - u - v)_q^{(a)} \tag{8}$$

We claim that this formal power series has, in addition to (1)-(3), the following properties:

$$(1 - q^{-b}v - u)_q^{(a)} (1 - v - q^{-a}u))_q^{(b)} = (1 - u - v)_q^{(a+b)}, \tag{9}$$

$$\left(1-w(1-q^{-a}v-q^{-1}u)^{-1}\right)_q^{(a)}(1-u-v)_q^{(a)}=(1-u-v-w)_q^{(a)}, \tag{10}$$

$$(1-u-v)_q^{(a)}\left(1-(1-q^{-1}v-q^{-a}u)^{-1}w\right)_q^{(a)}=(1-u-v-w))_q^{(a)} \tag{11}$$

where $vu=quv$ everywhere, $vw=qwv$ and $uw=q^{-1}wu$ in (10), (11). Putting $u=0$ or $v=0$ we get (1)-(3) as particular cases. The proof of (9)-(11) is based on the following observation:

$$(1-q^a v-u)(1-q^b v-q^{-1}u)=(1-q^b v-u)(1-q^a v-q^{-1}u) \tag{12}$$

for q-commuting variables v and u. Consider first (9). Note that it is enough to prove this identity for positive integers a and b only, because in this case both sides are finite power series and if they are equal for any q-commuting u, v, then their coefficients at ordered monomials are equal. But these coefficients are rational functions of q^a and q^b, so if they are equal for all positive integers a and b then they are equal identically.

From (1) we know, that for any positive integer n

$$(1-u)_q^{(n)}=(1-q^{-1}u)(1-q^{-2}u)\cdots(1-q^{-n}u) \tag{13}$$

Then we can reorder the factors of the product

$$(1-u-v)_q^{(n)}=(1-q^{-1}u-q^{-1}v)(1-q^{-2}u-q^{-2}v)\cdots(1-q^{-n}u-q^{-n}v).$$

using (12) and get another presentation:

$$(1-u-v)_q^{(n)}=(1-q^{-n}v-q^{-1}u)(1-q^{-(n-1)}v-q^{-2}u)\cdots(1-q^{-1}v-q^{-n}u). \tag{14}$$

From this presentation the relation (9) is obvious. Similarly we prove (10) for an integer positive a. Denote the left hand side by $F_a(u,v,w)$ and the right hand side of (10) by $G_a(u,v,w)$. We check first, that $F_1(u,v,w)=G_1(u,v,w)$. Next, we see from (9) that the function $F_n(u,v,w)$ satisfies the recurrence relation

$$F_{n+1}(u,v,w)=\left(1-q^{-1}w(1-q^{(n+1)}v-q^{-1}u)^{-1}\right)F_n(u,q^{-1}v,q^{-1}w)(1-q^{-1}v-q^{-n-1}u).$$

So, it remains to prove the same recurrence relation for $G_n(u,v,w)$ For this we note that we can, analogously to (14), prove the following identities:

$$(1-q^{-1}v-u-q^{-1}w)_q^{(n)}(1-q^{-1}v-q^{-n-1}u)=(1-q^{-n-1}v-q^{-1}u-q^{-2}w)$$
$$(1-q^{-n-2}v-q^{-2}u-q^{-3}w)\cdots(1-q^{-2}v-q^{-n}u-q^{-n-1}w)(1-q^{-1}v-q^{-n-1}u).$$

using the identity similar to (12)

$$(1-q^a v-q^b u-w)(1-q^{a+1}v-q^{b-1}u)=(1-q^a v-q^b u)(1-q^{a+1}v-q^{b-1}u-w)$$

Then we get:

$$\begin{aligned}&(1-q^{-1}v-u-q^{-1}w)_q^{(n)}(1-q^{-1}v-q^{-n-1}u)=(1-q^{-n-1}v-q^{-1}u)\\&(1-q^{-n-2}v-q^{-2}u-q^{-2}w)\cdots(1-q^{-1}v-q^{-n-1}u-q^{-n-1}w).\end{aligned}$$

The remaining part is straightforward.

We can get a rational degeneration of the identities above by the following procedure [1]: Put

$$x=u+\frac{\eta}{q^{-1}-1}v,\qquad y=v,\qquad z=w. \tag{15}$$

Then the q-commutativity relation $vu=quv$ transforms into

$$xy-q^{-1}yx=-\eta y^2 \tag{16}$$

and we can rewrite the equalities (9)-(11) in the variables x, y and z:

$$\begin{aligned}&(1-x-\eta(c)_{\overline{q}}y)_q^{(a+b)}=(1-x-\eta(c+b)_{\overline{q}}y)_q^{(a)}(1-q^{-a}x-q^{-a}\eta(c-a)_{\overline{q}}y)_q^{(b)}\\&\left(1-z(1-\overline{q}x-\eta\overline{q}(c+a-1)_{\overline{q}}y)^{-1}\right)_q^{(a)}(1-x-\eta(c)_{\overline{q}}y)_q^{(a)}=(1-x-\eta(c)_{\overline{q}}y-z)_q^{(a)},\\&(1-x-\eta(c)_{\overline{q}}y)_q^{(a)}\left(1-(1-q^{-a}x-\eta q^{-a}(c-a+1)_{\overline{q}}y)^{-1}z\right)_q^{(a)}=(1-x-\eta(c)_{\overline{q}}y-z)_q^{(a)}\end{aligned}$$

Here $\overline{q}=q^{-1}$, $xy-q^{-1}yx=-\eta y^2$, as before, $yz=qzy$, and $xz-q^{-1}zx=-\eta(2)_{\overline{q}}yz$. All the relations make sense for the Yangian limit $q=1$. In this case the q-power series becomes the usual geometric series for the power function $(1-x)^a$, which is considered now as a function of linear combinations of the Yangian variables x and y, $[x,y]=-\eta y^2$. The basic properties (9)-(11) can be rewritten as

$$(1-x-\eta cy)^{a+b}=(1-x-\eta(c+b)y)^a(1-x-\eta(c-a)y)^b \tag{17}$$

$$\left(1-z(1-x-\eta(c+a-1)y)^{-1}\right)^a(1-x-\eta cy)^a=(1-x-\eta cy-z)^a, \tag{18}$$

$$(1-x-\eta cy)^a\left(1-(1-x-\eta(c-a+1)y)^{-1}z\right)^a=(1-x-\eta cy-z)^a \tag{19}$$

where $[x,y]=-\eta y^2$, as before, $[y,z]=0$, and $[x,z]=-2\eta yz$.

3 Twisting cocycles

Let $e_{\pm\alpha}, e_{\pm(\delta-\alpha)}, q^{h_{\pm\alpha}} = q^{\pm h}$ be the generators of the quantum affine algebra $U_q(\widehat{sl}_2)$ with zero central charge, satisfying the relations:

$$q^{\pm h} e_{\pm\alpha} = q^{\pm 2} e_{\pm\alpha}, \qquad q^{\pm h} e_{\pm(\delta-\alpha)} = q^{\mp 2} e_{\pm\alpha},$$

$$[e_\alpha, e_{-\alpha}] = \frac{q^h - q^{-h}}{q - q^{-1}}, \qquad [e_{\delta-\alpha}, e_{-\delta+\alpha}] = \frac{q^{-h} - q^h}{q - q^{-1}}, \qquad [e_{\pm\alpha}, e_{\mp(\delta-\alpha)}] = 0,$$

plus q-Serre relations, which we do not use here. The comultiplication is given by the following formulas:

$$\begin{array}{cc} \Delta(e_\alpha) = e_\alpha \otimes 1 + q^{-h} \otimes e_\alpha, & \Delta(e_{\delta-\alpha}) = e_{\delta-\alpha} \otimes 1 + q^h \otimes e_{\delta-\alpha}, \\ \Delta(e_{-\alpha}) = e_{-\alpha} \otimes q^h + 1 \otimes e_{-\alpha}, & \Delta(e_{-\delta+\alpha}) = e_{-\delta+\alpha} \otimes q^{-h} + 1 \otimes e_{\delta-\alpha}. \end{array} \tag{20}$$

We claim that the element

$$\mathcal{F} = \left(1 - (2)_{q^2} \left(a \cdot 1 \otimes e_{\delta-\alpha} + b \cdot q^{-h} \otimes q^{-h} e_{-\alpha}\right)\right)_{q^2}^{\left(-\frac{h \otimes 1}{2}\right)}$$

satisfies the cocycle identity for any constants a and b.

Let us prove this statement. Set $u = (2)_{q^2} a e_{\delta-\alpha}, v = (2)_{q^2} b q^{-h} e_{-\alpha}$. Then $vu = q^2 uv$. We can rewrite the cocycle equation

$$\mathcal{F}_{12} (\Delta \otimes id) \mathcal{F} = \mathcal{F}_{23} (id \otimes \Delta) \mathcal{F},$$

using the tensor notations $a_1 = a \otimes 1 \otimes 1$, $a_2 = 1 \otimes 2 \otimes 1$, $a_3 = 1 \otimes 1 \otimes a$, as follows:

$$\begin{array}{c} \left(1 - q^{-h_1} v_2 - u_2\right)_{q^2}^{\left(-\frac{h_1}{2}\right)} \left(1 - q^{-h_1-h_2} v_3 - u_3\right)_{q^2}^{\left(-\frac{h_1+h_2}{2}\right)} = \\ \left(1 - q^{-h_2} v_3 - u_3\right)_{q^2}^{\left(-\frac{h_2}{2}\right)} \left(1 - q^{-h_1} v_2 - q^{-h_1-h_2} v_3 - u_2 - q^{h_2} u_3\right)_{q^2}^{\left(-\frac{h_1}{2}\right)} . \end{array} \tag{21}$$

Using (9) we see that we have to prove the following equality:

$$\begin{array}{c} \left(1 - v_3 - q^{h_2} u_3\right)_{q^2}^{\left(\frac{h_2}{2}\right)} \left(1 - q^{-h_1} v_2 - u_2\right)_{q^2}^{\left(-\frac{h_1}{2}\right)} \left(1 - q^{-h_1-h_2} v_3 - u_3\right)_{q^2}^{\left(-\frac{h_1+h_2}{2}\right)} = \\ \left(1 - q^{-h_1} v_2 - q^{-h_1-h_2} v_3 - u_2 - q^{h_2} u_3\right)_{q^2}^{\left(-\frac{h_1}{2}\right)} . \end{array} \tag{22}$$

Let us present the second factor of the left hand side of (22) as a series and permute the first factor with each term of this series. Then we get in the left hand side of (22):

$$\sum_{n \geq 0} C_n (q^{-h_1} v_2 + u_2)^n \left(1 - v_3 - q^{h_2 - 2n} u_3\right)_{q^2}^{\left(\frac{h_2}{2} - n\right)} \left(1 - q^{-h_1-h_2} v_3 - u_3\right)_{q^2}^{\left(-\frac{h_1+h_2}{2}\right)} \tag{23}$$

where $C_n = \frac{(-h_1/2)_{q^2}(-h_1/2+1)_{q^2}\cdots(-h_1/2+n-1)_{q^2}}{(n)_{q^2}!}$. Then, again using (9), we rewrite the LHS of (22) as

$$\begin{array}{c} \sum_{n\geq 0} C_n \left(q^{-h_1}v_2 + u_2\right)^n \left(1 - q^{-h_2}v_3 - q^{h_2-2n}u_3\right)_{q^2}^{(-n)} \\ \left(1 - v_3 - q^{h_2}u_3\right)_{q^2}^{\left(\frac{h_2}{2}\right)} \left(1 - q^{-h_1-h_2}v_3 - u_3\right)_{q^2}^{\left(-\frac{h_1+h_2}{2}\right)} . \end{array} \tag{24}$$

Repeating the factorization procedure for negative powers, we can present the product of the first two factors in (24) as a total (usual) power:

$$\left(q^{-h_1}v_2 + u_2\right)^n \left(1 - q^{-h_2}v_3 - q^{h_2-2n}u_3\right)_{q^2}^{(-n)} = \left((q^{-h_1}v_2 + u_2)(1 - q^{-h_2}v_3 - q^{h_2-2}u_3)^{-1}\right)^n .$$

Therefore the left hand side of (22) is equal to

$$\left(1 - (q^{-h_1}v_2 + u_2)(1 - q^{-h_2}v_3 - q^{h_2-2}u_3)^{-1}\right)_{q^2}^{\left(-\frac{h_1}{2}\right)} \left(1 - q^{-h_1-h_2}v_3 - q^{h_2}u_3\right)_{q^2}^{\left(-\frac{h_1}{2}\right)}$$

One can see, that the desired equality (22) is now precisely the generalized Faddeev-Volkov identity (10).

Further, as in the previous section, we can make a change of variables, see [11]:

$$f_1 = e_{\delta-\alpha} + \frac{\eta}{q^{-2}-1} q^{-h} e_{-\alpha}, \qquad f_0 = q^{-h} e_{-\alpha}. \tag{25}$$

The elements f_1, f_0 and h generate a Hopf subalgebra of $U_q(\widehat{sl}_2)$, considered now [11] as an algebra over $\mathbb{C}[[\eta]](q)$:

$$[h, f_1] = -2f_1, \qquad [h, f_0] = -2f_0, \qquad f_1 f_0 - q^{-2} f_0 f_1 = -\eta f_0^2, \tag{26}$$

$$\Delta(f_0) = f_0 \otimes 1 + q^{-h} \otimes f_0, \qquad \Delta(f_1) = f_1 \otimes 1 + q^h \otimes f_1 + \eta q^h (h)_{q^{-2}} \otimes f_0. \tag{27}$$

Then the twisting element $\mathcal{F}$ after a proper normalization of the constants a and b, $a = \xi, b = \frac{\eta}{q^{-2}-1}$, has the form

$$\mathcal{F} = \Big(1 - (2)_{q^2}\xi(1 \otimes f_1 + \eta\, (h/2)_{q^{-2}} \otimes f_0)\Big)_{q^2}^{\left(-\frac{h\otimes 1}{2}\right)} \tag{28}$$

Again, it makes sense in the Yangian limit $q = 1$ where $\mathcal{F}$ has the following form:

$$\mathcal{F} = \left(1 - 2\xi(1 \otimes f_1 + \eta\frac{h}{2} \otimes f_0)\right)^{-\frac{h\otimes 1}{2}} . \tag{29}$$

4 Twisted R-matrices and deformed Hamiltonians

Let $\pi_{1/2}(z)$ be the two dimensional vector representation of the algebra $U_q(\widehat{sl}_2)$. In this representation the generator $e_{-\alpha}$ acts as a matrix unit e_{21}, $e_{\delta-\alpha}$ as ze_{21} and h as $e_{11}-e_{22}$. The R-matrix in the tensor product $\pi_{1/2}(z_1)\otimes\pi_{1/2}(z_2)$ of $U_q(\widehat{sl}_2)$ is well known. For the comultiplication (20) it is

$$\begin{gathered}R_0(z_1,z_2)=e_{11}\otimes e_{11}+e_{22}\otimes e_{22}+\tfrac{z_1-z_2}{q^{-1}z_1-qz_2}\left(e_{11}\otimes e_{22}+e_{22}\otimes e_{11}\right)+\\ \tfrac{q^{-1}-q}{q^{-1}z_1-qz_2}\left(z_2e_{12}\otimes e_{21}+z_1e_{21}\otimes e_{12}\right),\end{gathered}\tag{30}$$

$$R_0(z_1,z_2)=\begin{pmatrix}1&0&0&0\\ 0&\frac{z_1-z_2}{q^{-1}z_1-qz_2}&\frac{(q^{-1}-q)z_2}{q^{-1}z_1-qz_2}&0\\ 0&\frac{(q^{-1}-q)z_1}{q^{-1}z_1-qz_2}&\frac{z_1-z_2}{q^{-1}z_1-qz_2}&0\\ 0&0&0&1\end{pmatrix}\tag{31}$$

The image of the element $\mathcal{F}$ has the form

$$F=1+\frac{q^h-1}{q-1}\left(az_2+bq^{-h+1}\right)\otimes e_{21}=1+((az_2+b)e_{11}-(q^{-1}az_2+qb)e_{22})\otimes e_{21},$$

Hence, the twisted R-matrix $R^F=F^{21}RF^{-1}$ can be written as

$$\begin{gathered}R^F(z_1,z_2)=R_0(z_1,z_2)+\tfrac{z_1-z_2}{q^{-1}z_1-qz_2}((b+az_2)(e_{22}-e_{11})\otimes e_{21}+\\ +(q^{-1}az_1+qb)e_{21}\otimes(e_{11}-e_{22})+(b+az_2)(q^{-1}az_1+qb)e_{21}\otimes e_{21}),\end{gathered}\tag{32}$$

$$R^F(z_1,z_2)=\tfrac{z_1-z_2}{q^{-1}z_1-qz_2}\begin{pmatrix}\frac{q^{-1}z_1-qz_2}{z_1-z_2}&0&0&0\\ -(az_2+b)&1&\frac{(q^{-1}-q)z_2}{z_1-z_2}&0\\ q^{-1}az_1+qb&\frac{(q^{-1}-q)z_1}{z_1-z_2}&1&0\\ (az_2+b)(q^{-1}az_1+qb)&-(q^{-1}az_1+qb)&az_2+b&\frac{q^{-1}z_1-qz_2}{z_1-z_2}\end{pmatrix}$$

It satisfies the basic property $R^F(z,z)=P_{12}$, where P_{12} is a permutation of the tensor factors. Let $t(z)=Tr_0R^F_{0N}(z,z_2)R^F_{0,N-1}(z,z_2)\cdots R^F_{01}(z,z_2)$ be a family of commuting transfer matrices for the corresponding periodic chain, $[t(z'),t(z'')]=0$ (where we treat z_2 is a parameter of the theory and $z=z_1$ is considered as a spectral parameter). Then the Hamiltonian

$$H_{a,b,z_2}=(q^{-1}-q)z\frac{d}{dz}t(z)|_{z=z_2}t^{-1}(z_2)$$

can be computed by a standard procedure,

$$H_{a,b,z_2} = (q^{-1} - q) \sum_k P_{k,k+1} z \frac{d}{dz} R_{k,k+1}(z, z_2)|_{z=z_2},$$

and is equal to

$$H_{a,b,z_2} = H_{XXZ} + \sum_k \left(C \left(\sigma_k^z \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^z \right) + D \sigma_k^- \sigma_{k+1}^- \right) \tag{33}$$

here $C = \frac{q-1}{2}(b - az_2 q^{-1})$, $D = (az_2 + b)(q^{-1}az_2 + qb)$; $\sigma^+ = e_{12}$, $\sigma^- = e_{21}$, $\sigma^z = e_{11} - e_{22}$ and

$$H_{XXZ} = \sum_k \left(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + \frac{q + q^{-1}}{2} \sigma_k^z \sigma_{k+1}^z \right). \tag{34}$$

We see that by a suitable choice of the parameters a, b, z_2 we can add to the XXZ Hamiltonian arbitrary linear combination of the terms $\sum_k \sigma_k^z \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^z$ and $\sum_k \sigma_k^- \sigma_{k+1}^-$ and the model will remain integrable.

In order to get the corresponding XXX degeneration and, moreover, to have a unified description of both models, we use again the realization (25) of $U_q(\widehat{sl}_2)$ and the evaluation homomorphism $\pi_{1/2}(u)(f_1) = \left(u + \eta\, (h/2)_{q^2} \right) q^h f_0$, $\pi_{1/2}(u)(f_0) = q\sigma^-$, which effectively corresponds to a shift of spectral parameter $z = u - \frac{\eta}{q^{-2}-1}$. In these notations the non-twisted R-matrix $R_0(u_1, u_2)$ has the form

$$R_0(u_1,u_2) = \tfrac{1}{2}(1 + \sigma^z \otimes \sigma^z) + \tfrac{u_1 - u_2}{2(q^{-1}u_1 - qu_2 - q\eta)}(1 - \sigma^z \otimes \sigma^z) + \\ \tfrac{(q^{-1}-q)u_2 - q\eta}{q^{-1}u_1 - qu_2 - q\eta} \sigma^+ \otimes \sigma^- + \tfrac{(q^{-1}-q)u_1 - q\eta}{q^{-1}u_1 - qu_2 - q\eta} \sigma^- \otimes \sigma^+,$$

and the twisted R-matrix $R^F(u_1, u_2)$ is equal to

$$R^F(u_1,u_2) = R_0(u_1,u_2) + \tfrac{u_1 - u_2}{q^{-1}u_1 - qu_2 - q\eta}\big(-\xi u_2 \sigma^z \otimes \sigma^- + \\ + \xi(q^{-1}u_1 - q\eta)\sigma^- \otimes \sigma^z + \xi^2 u_2 (q^{-1}u_1 - q\eta)\sigma^- \otimes \sigma^- \big), \tag{35}$$

$$R^F(u_1,u_2) = \tfrac{u_1 - u_2}{q^{-1}u_1 - qu_2 - q\eta} \begin{pmatrix} \frac{q^{-1}u_1 - qu_2 - q\eta}{u_1 - u_2} & 0 & 0 & 0 \\ -\xi u_2 & 1 & \frac{(q^{-1}-q)u_2 - q\eta}{u_1 - u_2} & 0 \\ \xi(q^{-1}u_1 - q\eta) & \frac{(q^{-1}-q)u_1 - q\eta}{u_1 - u_2} & 1 & 0 \\ \xi^2 u_2 (q^{-1}u_1 - q\eta) & -\xi(q^{-1}u_1 - q\eta) & \xi u_2 & \frac{q^{-1}u_1 - qu_2 - q\eta}{u_1 - u_2} \end{pmatrix}$$

In particular, for $q=1$ we get a deformation of the Yang R-matrix:

$$R^F(u_1,u_2) = \tfrac{u_1-u_2}{u_1-u_2-\eta}(1-\eta\tfrac{P_{12}}{u_1-u_2} - \\ \xi u_2\sigma^z\otimes\sigma^- + \xi(u_1-\eta)\sigma^-\otimes\sigma^z + \xi^2 u_2(u_1-\eta)\sigma^z\otimes\sigma^z), \tag{36}$$

Again, the R-matrix $R^F(u_1,u_2)$ satisfies the property $R^F(u,u)=P_{12}$, and the Hamiltonian

$$H_{\eta,\xi,u_2} = ((q^{-1}-q)u - q^{-1}\eta)\frac{d}{du}t(u)|_{u=u_2}t^{-1}(u_2)$$

for $t(u) = Tr_0 R^F_{0N}(u,u_2)R^F_{0,N-1}(u,u_2)\cdots R^F_{01}(u,u_2)$ is given by the same formula (33), where $C=\xi\frac{q^{-1}-1}{2}u_2 - \frac{q^{-1}\xi\eta}{2}, D=\xi^2u_2(q^{-1}u_2-q\eta)$ and now also makes sense in XXX limit $q=1$,

$$H_{\eta,\xi,u_2} = H_{XXX} + \sum_k \left(C\left(\sigma^z_k\sigma^-_{k+1}+\sigma^-_k\sigma^z_{k+1}\right) + D\sigma^-_k\sigma^-_{k+1}\right) \tag{37}$$

where $C=-\frac{\xi\eta}{2}$, $D=\xi^2u_2(u_2-\eta)$.

5 Discussions

1. One can see that the R-matrix (32) is a quantization of the following solution of the classical YB equation:

$$r_{a,b}(z_1,z_2) = r_{DJ}(z_1,z_2) + a(z_1\sigma^-\otimes\sigma^z - z_2\sigma^z\otimes\sigma^-) + b(\sigma^-\otimes\sigma^z - \sigma^z\otimes\sigma^-) \tag{38}$$

where $r_{DJ}(z_1,z_2) = \frac{1}{2}\left(\frac{z_1+z_2}{z_1-z_2}t_{12} - \sigma^+\otimes\sigma^- + \sigma^-\otimes\sigma^+\right)$ is the Drinfeld-Jimbo solution of the classical YB equation. Here t_{12} is the splitted Casimir operator, $t_{12} = \sigma^-\otimes\sigma^+ + \sigma^+\otimes\sigma^- + \frac{1}{2}\sigma^z\otimes\sigma^z$. The r-matrix (38) is gauge equivalent to

$$\tilde{r}_{a,b}(z_1,z_2) = r_{DJ}(z_1,z_2) + a(z_1\sigma^-\otimes\sigma^z - z_2\sigma^z\otimes\sigma^-) + 4ab(z_1-z_2)(\sigma^-\otimes\sigma^-) \tag{39}$$

The gauge equivalence is given by $Ad(1+2b\sigma^-)\otimes Ad(1+2b\sigma^-)$. It can be shown that for generic a and b the r-matrix (39) is gauge equivalent to the following solution of the YB equation found in [2] (see [3] for the quantum version):

$$r_{BD}(z_1,z_2) = r_{DJ}(z_1,z_2) + (z_1-z_2)(\sigma^-\otimes\sigma^-). \tag{40}$$

Therefore in the case of sl_2 we have a description of the quantization of all the trigonometric solutions of the YB equation, described in [2], in the universal form. Moreover, the rational degeneration (36) is a quantization of the rational r-matrix found in [10]:

$$r_{St}(u_1,u_2) = \frac{t_{12}}{u_1-u_2} + \xi(u_1\sigma^- \otimes \sigma^z - u_2\sigma^z \otimes \sigma^-),$$

and thus we answer the similar question of a quantization of the rational sl_2 solutions of the classical YB equation (see also [7]).

2. It will be interesting to study the spectra and the eigenstates of the Hamiltonians (33), (37). The particular case of (37) with $C = 0$ was studied in [9]. The study was based on a quantization of a more simple rational r-matrix, suggested in [7]. It was shown, that in this case the spectrum of the Hamiltonian remains unchanged after the deformation. However the deformed Hamiltonian has Jordanian blocks and thus it is not diagonalizable. Therefore we can expect that at least the deformed XXX chains (37) are not equivalent to the undeformed one.

3. We see that it turned out to be very important to obtain a two-parameter deformation of the algebra $U_q(\widehat{sl}_2)$ and of its fundamental R-matrix. Only in such a way we managed to get the deformation of the Yangian $Y(sl_2)$, the corresponding rational R-matrix (36) and the related Hamiltonian (37). On the classical level, the generic r-matrices of this family are gauge equivalent. It is interesting to understand, whether these equivalences can be extended to the quantum level and to develop the representation theory of the corresponding deformed two-parameter Hopf algebra.

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